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LETTER TO THE EDITOR

Bell's inequality for mixed states

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Abstract. Violation of Bell's inequality for mixed states is considered. We show that a mixture of states that are macroscopically essentially equivalent may suppress the violation. Thus in general the inequality will not be violated by mixed states (except very special ones).

Bell [1] demonstrated that quantum mechanics is a theory that violates local realism. His proof involves violation of certain inequalities, thus the extent of the violation may provide some measure for the non-locality of the theory. In this letter we shall confine ourselves to the particular inequality (equation (4) below) that was derived in [2]. It has been shown [3, 4] that all entangled (i.e. non-product) pure states can, by appropriate choice of observables, lead to a violation of Bell's inequality (BIQ). Similarly and trivially it was shown [4] that product states cannot lead to such violation. The above results are qualitative, i.e. they are applicable to states made up of particles of arbitrary spin. Thus it was shown [5, 6] that two particles each with high (macroscopic) spin may still lead to maximal violation of BIQ. We shall show below that mixing such a state with states that are equivalent macroscopically (but microscopically different) renders the violation of BIQ unobservable. Thereby we show that although quantum mechanics violates local realism this very theory restricts the occurrence of this bizarre attribute.

This letter is organized as follows. We briefly outline the proof [4, 5] that every pure entangled state may lead to violation of BIQ. Then we show that a certain class of mixed states whose index of correlation [7] is non-zero cannot lead to violation of BIQ. These we term states possessing classical-like correlations. Following this we consider an example of a mixed state which cannot violate BIQ even though each of its two constituent pure states violates BIQ maximally. At this juncture we present our main point: a mixture of two pure states which are macroscopically indistinguishable will not violate the inequality though each state can lead to maximal violation of BIQ.

We now show that every entangled state can lead to violation of BIQ. Let $\mathcal{H}_C$ ($\mathcal{H}_D$) be the Hilbert space pertaining to C (D). Let the normalized entangled state be

$$|\Psi\rangle = \sum_n a_n |\psi_n\rangle|\phi_n\rangle. \qquad (1)$$

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Here $\{|\psi_n\rangle\}$ is an orthonormal base of $\mathcal{H}_C$, and $\{|\phi_n\rangle\}$ is an orthonormal base of $\mathcal{H}_D$, and we take $a_1 \neq 0$, $a_2 \neq 0$. (We may take a_n as positive real numbers in the Schmidt decomposition, equation (1).) Let us define the Hermitian operators (whose eigenvalues are ± 1)

$$C(\gamma) = I_C - 2|e_\gamma\rangle\langle e_\gamma| \tag{2a}$$

$$D(\delta) = I_D - 2|e_\delta\rangle\langle e_\delta|. \tag{2b}$$

Here $|e_\gamma\rangle$ is a unit vector in $\mathcal{H}_C$ such that

$$\langle e_\gamma \mid \psi_n\rangle = 0 \qquad \text{for} \quad n > 2 \tag{3}$$

with a similar relation for $|e_\delta\rangle$ and ϕ_n. I_C is the unit operator in $\mathcal{H}_C$; I_D plays a similar role in $\mathcal{H}_D$. We refer to $C(\gamma)$ and $D(\delta)$ as our 'observables' and to γ and δ as the 'orientations' of the apparatus. We search for orientations leading to violation of BIQ. For the latter we take [2]

$$|\langle C(\gamma)D(\delta)\rangle + \langle C(\gamma')D(\delta)\rangle + \langle C(\gamma)D(\delta')\rangle - \langle C(\gamma')D(\delta')\rangle| \equiv f \leqslant 2. \tag{4}$$

Defining

$$\langle e_\gamma \mid \psi_1\rangle = \cos(\gamma/2) \qquad \langle e_\gamma \mid \psi_2\rangle = \sin(\gamma/2) \tag{5a}$$

$$\langle e_\delta \mid \phi_1\rangle = \cos(\delta/2) \qquad \langle e_\delta \mid \phi_2\rangle = \sin(\delta/2) \tag{5b}$$

one readily obtains

$$\langle\Psi \mid C(\gamma)D(\delta) \mid \Psi\rangle = 1 - (a_1^2 + a_2^2) + (a_1^2 + a_2^2)\cos\gamma\cos\delta + 2a_1a_2\sin\gamma\sin\delta. \tag{6}$$

Substituting this, and the corresponding results for γ' and δ' in equation (4), and choosing $\gamma = 0$, $\gamma' = \pi/2$, $\delta = -\delta' = \arctan c$, with $c = 2a_1a_2/(a_1^2 + a_2^2)$ we get

$$f = 2 + 2(a_1^2 + a_2^2)(\sqrt{1 + c^2} - 1) > 2. \tag{7}$$

These values for the orientations were chosen to maximize f as a function of the orientation angles. Clearly for normalized states with only two non-zero terms (i.e. $a_1^2 + a_2^2 = 1$) maximal violation is obtained for $a_1 = a_2 = 1/\sqrt{2}$. This is the maximal violation attainable [8].

It is easy to see [4] that a pure non-entangled state (i.e. a product state) cannot violate the BIQ. Similarly it can be seen that whenever the density matrix in the combined spaces is a product no violation of BIQ is possible. We note that in this case, viz. $\rho_{CD} = \rho_C \otimes \rho_D$, the index of correlation of the system [7] vanishes. We now show that a density matrix whose diagonal form is $(1 \geqslant \rho_n \geqslant 0, \sum \rho_n = 1)$

$$\rho_{CD} = \sum_n \rho_n|\psi_n\rangle|\phi_n\rangle\langle\phi_n|\langle\psi_n| \tag{8}$$

which, in general, cannot be written as a product (i.e. $\rho_{CD} \neq \rho_C \otimes \rho_D$ and thus its index of correlation does not vanish [7]) cannot lead to violation of BIQ. The proof is immediate upon the substitution of this state (equation (8)) into the LHS of equation

(4) and noting that the expectation value splits into a sum of terms each weighed with ρ_n and involving a product state which cannot violate the BIQ. Thus this case, i.e. ρ_{CD} of the form of equation (8), can be associated with correlations (between C and D) which are classical-like.

We now turn to the more interesting case. We consider a mixed state whose diagonal form is

$$\rho_{CD} = \sum_n \rho_n |\Psi_n\rangle\langle\Psi_n|. \tag{9}$$

Here at least one state $|\Psi_n\rangle$ is entangled. In particular we consider two particles, C and D, with total spin j_C, j_D respectively. We now consider a special case $\rho_n = 0$ for $n > 2$, $\rho_n \neq 0$ for $n \leqslant 2$, and

$$|\Psi_1\rangle = |j_C, m_C\rangle|j_D, m_D\rangle \tag{10a}$$

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}[|j_C, m_C - 1\rangle|j_D, m_D\rangle + |j_C, m_C\rangle|j_D, m_D - 1\rangle]. \tag{10b}$$

Clearly for the second state maximal violation of BIQ can be achieved (cf. equation (7)). Here our relevant C space is again two dimensional $(|j_C, m_C\rangle, |j_C, m_C - 1\rangle)$. So is the D space. We may again apply the definitions (equations (5)), with $\langle e_\gamma \mid j_C, m_C - 1\rangle = \cos(\gamma/2)$, $\langle e_\delta \mid j_D, m_D\rangle = \cos(\delta/2)$, $\langle e_\gamma \mid j_C, m_C\rangle = \sin(\gamma/2)$, $\langle e_\delta \mid j_D, m_D - 1\rangle = \sin(\delta/2)$. A straightforward calculation leads to

$$\langle C(\gamma)D(\delta)\rangle = (\rho_2 - \rho_1)\cos\gamma\cos\delta + \rho_2 \sin\gamma\sin\delta. \tag{11}$$

Thus for $\rho_1 = \rho_2$ no violation of BIQ is possible. More generally, the maximum of f is now $2\sqrt{\rho_2 - \rho_1)^2 + \rho_2^2}$, which decreases from $2\sqrt{2}$ to 2 when ρ_1 increases from 0 to $1/5$. Hence, for $\rho_1 \geqslant 1/5$ BIQ cannot be violated. If one considers the more complicated case where both $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are maximally entangled states, it is easy to show that for the mixed state the violation is, if at all, generally smaller. An illustrative extremal case is when all entangled states (equation 9) are equally probable—then a simple rearrangement of terms leads to a classical-like density matrix (equation (8)).

Consider now the case wherein, in equations (10), $j_C \simeq j_D \simeq 10^9$ (i.e. a large value) and $m_C \simeq m_D \simeq j_C$. In this case the two states of equations (10) are, from the macroscopic viewpoint, indistinct. Thus we see that a macroscopic state which is a mixture of microscopically distinct states will not in general lead to violation of BIQ. If the violation is the measure of non-locality, we see that non-locality is not expected to be observed for macroscopic mixed states—bar mixing of very particular states only.

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